14) Appendix III: Spherical Vector Field Solutions

From the characteristic solutions of the scalar wave equation $\nabla^2\psi + k^2\psi = 0$ in spherical coordinates

$$\boxed{f_{emn} = \cos m\varphi \; P_n^m(\cos\theta)\, z_n(kr) \qquad f_{omn} = \sin m\varphi \; P_n^m(\cos\theta)\, z_n(kr)}$$

we can build vector field solutions of the equation $\vec{\nabla}(\vec{\nabla}\cdot\vec{C}) - \vec{\nabla}\times\vec{\nabla}\times\vec{C} + k^2\vec{C} = \vec{0}$

$$\vec{L} = \vec{\nabla}\psi \qquad \vec{M} = \vec{\nabla}\times\psi\vec{a} = \vec{L}\times\vec{a} \qquad \vec{N} = \frac{1}{k}\vec{\nabla}\times\vec{M}$$

Problem! $\vec{a}$ must be a constant vector and then $\vec{M}$ and $\vec{N}$ will never be purely normal or tangential over the entire surface of spheres centered at the origin: not good to write boundary conditions in spherically symetric problems! We would need $\vec{a} \sim \vec{e}_r$

Fortunately, it turns out to be possible to construct $\vec{M}$ and $\vec{N}$ as

$$\boxed{\vec{L} = \vec{\nabla}\psi \qquad \vec{M} = \vec{\nabla}\times\vec{r}\psi = \vec{L}\times\vec{r} \qquad \vec{N} = \frac{1}{k}\vec{\nabla}\times\vec{N}}$$

Proof: Let us try to find a solution of $\vec{\nabla}(\vec{\nabla}.\vec{C}) - \vec{\nabla}\times\vec{\nabla}\times\vec{C} + k_0^2\vec{C} = \vec{0}$

in the form $\vec{M} = \vec{\nabla}\times(u(r)\vec{e_r}\psi) = \vec{L}\times u(r)\vec{e_r} + \psi\underbrace{\vec{\nabla}\times u(r)\vec{e_r}}_{0}$

Since $\vec{L} = \vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\vec{e_\theta} + \frac{1}{rs\theta}\frac{\partial\psi}{\partial\varphi}\vec{e_\varphi}$

$$\vec{M} = \begin{pmatrix} \partial_r\psi \\ \frac{1}{r}\partial_\theta\psi \\ \frac{1}{rs\theta}\partial_\varphi\psi \end{pmatrix} \times \begin{pmatrix} u(r) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{rs\theta}\partial_\varphi\psi\, u(r) \\ -\frac{1}{r}\partial_\theta\psi\, u(r) \end{pmatrix}$$

$$\Rightarrow \vec{M} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(u\psi)\vec{e_\theta} - \frac{1}{r}\frac{\partial}{\partial\varphi}(u\psi)\vec{e_\varphi}$$

$\vec{\nabla}.\vec{M} = 0$

$\vec{\nabla}\times\vec{\nabla}\times\vec{M} - k^2\vec{M}$ can be expressed in spherical coordinates (exercise). The $\vec{e_r}$ component is 0 and the $\vec{e_\theta}$ and $\vec{e_\varphi}$ components are identical and satisfied only if $u(r) = r$. QED

We therefore have :

$$\boxed{\begin{aligned} \vec{M} &= \frac{1}{\sin\theta}\frac{\partial\psi}{\partial\varphi}\vec{e_\theta} - \frac{\partial\psi}{\partial\theta}\vec{e_\varphi} \\ \vec{N} &= \left[\frac{1}{k}\frac{\partial^2(r\psi)}{\partial r^2} + kr\psi\right]\vec{e_r} + \frac{1}{kr}\frac{\partial^2(r\psi)}{\partial r\partial\theta}\vec{e_\theta} + \frac{1}{kr\sin\theta}\frac{\partial^2(r\psi)}{\partial r\partial\varphi}\vec{e_\varphi} \end{aligned}}$$

[[Proof: Exercise]]

Since $\psi = f_1(r) f_2(\theta) f_3(\varphi)$ is such that f_1 (and hence ψ) satisfies

$$r^2 \frac{d^2 f_1}{dr^2} + 2r \frac{df_1}{dr} + \left(k^2 r^2 - n(n+1)\right) f_1 = 0$$

[eq (1) in the note on scalar spherical waves]

the radial component of $\vec{N}$, $N_r = \frac{1}{k} \frac{\partial^2 (r\psi)}{\partial r^2} + k^2 r \psi$, reduces to

$$\boxed{N_r = \frac{n(n+1)}{kr} \psi}$$

<u>proof</u> : exercise ∎

Carrying out the differentiation for f_{emn} and f_{omn}, we get

$$\vec{M}_{\substack{e\\o}mn} = \left[\mp \frac{m}{\sin\theta} z_n(kr) P_n^m(\cos\theta) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(m\varphi)\right]\vec{e_\theta}$$
$$- \left[z_n(kr)\frac{\partial}{\partial\theta} P_n^m(\cos\theta) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(m\varphi)\right]\vec{e_\varphi}$$

$$\vec{N}_{\substack{e\\o}mn} = \left[\frac{n(n+1)}{kr} z_n(kr) P_n^m(\cos\theta) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(m\varphi)\right]\vec{e_r}$$
$$+ \left[\frac{1}{kr}\frac{\partial}{\partial r}\left[r\, z_n(kr)\right]\frac{\partial}{\partial\theta} P_n^m(\cos\theta) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(m\varphi)\right]\vec{e_\theta}$$
$$\mp \left[\frac{m}{kr\sin\theta}\frac{\partial}{\partial r}\left[r\, z_n(kr)\right] P_n^m(\cos\theta) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(m\varphi)\right]\vec{e_\varphi}$$

$$\vec{L}_{\substack{e\\o}mn} = \left[\frac{\partial}{\partial r} z_n(kr) P_n^m(\cos\theta) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(m\varphi)\right]\vec{e_r}$$
$$+ \left[\frac{1}{r} z_n(kr)\frac{\partial}{\partial\theta} P_n^m(\cos\theta) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(m\varphi)\right]\vec{e_\theta}$$
$$\mp \left[\frac{m}{r\sin\theta} z_n(kr) P_n^m(\cos\theta) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(m\varphi)\right]\vec{e_\varphi}$$

2. Orthogonality relations

We integrate over a sphere of radius r the scalar product between two spherical vector field solutions. $\vec{L}.\vec{M}$ & $\vec{M}.\vec{N}$ always orthogonal, same with even/odd, $n \neq n'$ or $m \neq m'$ combinations. The only non-zero scalar products are:

$$\int_0^{2\pi}\int_0^{\pi} \vec{L}_{\substack{e\\o}mn} \cdot \vec{L}_{\substack{e\\o}mn} \sin\theta \, d\theta \, d\varphi = (1+\delta)\frac{2\pi}{(2n+1)^2}\frac{(n+m)!}{(n-m)!}k^2\left\{n\left[z_{n-1}(kr)\right]^2 + (n+1)\left[z_{n+1}(kr)\right]^2\right\}$$

$$\int_0^{2\pi}\int_0^{\pi} \vec{M}_{\substack{e\\o}mn} \cdot \vec{M}_{\substack{e\\o}mn} \sin\theta \, d\theta \, d\varphi = (1+\delta)\frac{2\pi}{2n+1}\frac{(n+m)!}{(n-m)!}n(n+1)\left[z_n(kr)\right]^2$$

$$\int_0^{2\pi}\int_0^{\pi} \vec{N}_{\substack{e\\o}mn} \cdot \vec{N}_{\substack{e\\o}mn} \sin\theta \, d\theta \, d\varphi = (1+\delta)\frac{2\pi}{(2n+1)^2}\frac{(n+m)!}{(n-m)!}n(n+1)\left\{(n+1)\left[z_{n-1}(kr)\right]^2 + n\left[z_{n+1}(kr)\right]^2\right\}$$

$$\int_0^{2\pi}\int_0^{\pi} \vec{L}_{\substack{e\\o}mn} \cdot \vec{N}_{\substack{e\\o}mn} \sin\theta \, d\theta \, d\varphi = (1+\delta)\frac{2\pi}{(2n+1)^2}\frac{(n+m)!}{(n-m)!}n(n+1)k\left\{\left[z_{n-1}(kr)\right]^2 - \left[z_{n+1}(kr)\right]^2\right\}$$

with $\delta = 0$ if $m>0$ and $\delta = 1$ if $m=0$

3. Vector plane wave expansion

Let $\vec{f}(z)$ be a plane wave propagating along z and linearly polarized along $\vec{a} = \vec{e_x}, \vec{e_y}$ or $\vec{e_z}$.

$$\vec{f}(z) = \vec{a}\, e^{-jkz} = \vec{a}\, e^{-jkr\cos\theta}$$

We have :

$$\boxed{\begin{aligned}
\vec{e_x}\, e^{-jkz} &= \sum_{n=1}^{+\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[\vec{M}_{o1n} + j\, \vec{N}_{e1n}\right] \\
\vec{e_y}\, e^{-jkz} &= -\sum_{n=1}^{+\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[\vec{M}_{e1n} - j\, \vec{N}_{o1n}\right] \\
\vec{e_z}\, e^{-jkz} &= \frac{1}{k} \sum_{n=0}^{+\infty} (-j)^{n-1} (2n+1)\, \vec{L}_{e0n}
\end{aligned}}$$

where $\vec{M}, \vec{N}, \vec{L}$ are built from $\boxed{z_n = j_n}$